MATH 245 F19, Exam 1 Solutions

1. Carefully define the following terms: prime, Double Negation Theorem, Conditional Interpretation Theorem.

   Let $n \in \mathbb{N}$ with $n \geq 2$. If there is NO $a \in \mathbb{N}$ with $1 < a < n$ and $a|n$, then we call n prime. The Double Negation Theorem states: Let p be a proposition. Then $p \equiv \neg\neg p$. The Conditional Interpretation Theorem states: Let p, q be propositions. Then $p \to q \equiv q \vee \neg p$.

2. Carefully define the following terms: Addition Semantic Theorem, Vacuous Proof Theorem, contrapositive

   The Addition Semantic Theorem states: Let p, q be propositions. $p \vdash p \vee q$. The Vacuous Proof Theorem states: Let p, q be propositions. $(\neg p) \vdash (p \to q)$. The contrapositive of the compound proposition $p \to q$ is the proposition $(\neg q) \to (\neg p)$.

3. Let p, q be propositions. Prove or disprove: $(p \wedge q) \to (p \to q)$ is a tautology.

   The statement is true. Because the fifth column of the truth table (to the right) is all T, the proposition $(p \wedge q) \to (p \to q)$ is a tautology.

p	q	$p \wedge q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

4. Let $m, n \in \mathbb{Z}$. Prove or disprove: If $m|n$, then $m|n^2$.

   The statement is true. Suppose that $m|n$. Then there is some $s \in \mathbb{Z}$ with $ms = n$. Multiplying both sides by n, we have $m(sn) = n^2$. Since $sn \in \mathbb{Z}$ (being the product of two integers), we must have $m|n^2$.

5. Let $m, n \in \mathbb{N}_0$ with $n \geq m$. Evaluate and fully simplify $\frac{\binom{n+1}{m}}{\binom{n}{m}}$.

   We have $\frac{\binom{n+1}{m}}{\binom{n}{m}} = \frac{\frac{(n+1)!}{m!(n+1-m)!}}{\frac{n!}{m!(n-m)!}} = \frac{(n+1)!m!(n-m)!}{m!(n+1-m)!n!} = \frac{(n+1)n!m!(n-m)!}{m!(n+1-m)(n-m)!n!} = \frac{n+1}{n+1-m}$. If desired, this can be also written as $\frac{n+1-m+m}{n+1-m} = 1 + \frac{m}{n+1-m}$.

6. Prove or disprove: For arbitrary $x, y \in \mathbb{R}$, if x, y are both rational, then $\frac{x+y}{2}$ is rational.

   The statement is true, and we give a direct proof. We assume that x, y are rational. Hence there are integers a, b, c, d, with $b, d \neq 0$, such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$. Now, $\frac{x+y}{2} = \frac{\frac{a}{b}+\frac{c}{d}}{2} = \frac{ad+bc}{2bd}$. Now, $ad+bc, 2bd$ are both integers, and $2bd \neq 0$, so $\frac{x+y}{2}$ is rational.

7. Fix our domain to be $\mathbb{R}$. Simplify the proposition $\neg(\forall x\ \exists y\ \forall z,\ x < y \leq z)$ as much as possible (where nothing is negated).

   We begin by pulling $\neg$ inside the quantifiers, getting $\exists x\ \forall y\ \exists z\ \neg(x < y \leq z)$. Note that $x < y \leq z \equiv (x < y) \wedge (y \leq z)$, so we apply De Morgan's law to get $\exists x\ \forall y\ \exists z\ (\neg(x < y)) \vee (\neg(y \leq z))$. Lastly, we simplify the inequalities to get $\exists x\ \forall y\ \exists z\ (x \geq y) \vee (y > z)$. Note that this can NOT be written as a double inequality.

8. State and prove modus ponens, using semantic theorems only (no truth tables).

Thm: Let p, q be propositions. Then $p \to q, p \vdash q$.
Pf 1: We assume $p \to q$ and p. By conditional interpretation, $q \vee \neg p$. By double negation, $\neg\neg p$. By disjunctive syllogism, q.
Pf 2: We assume $p \to q$ and p. We have $p \to q \equiv (\neg q) \to (\neg p)$, its contrapositive. By double negation, $\neg\neg p$. By modus tollens, $\neg\neg q$. By double negation again, q.

9. Prove or disprove the proposition $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N}, x^2 < y < (x+1)^2$.

The statement is true. Let $x \in \mathbb{N}$ be arbitrary. Take $y = x^2 + 1$. Note that $y - x^2 = 1 \in \mathbb{N}_0$, so $x^2 \leq y$. But also $x^2 \neq y$, since $y - x^2 \neq 0$. Hence $x^2 < y$. Now, $(x+1)^2 - y = (x^2 + 2x + 1) - (x^2 + 1) = 2x$. Since $x \in \mathbb{N}$, $2x \in \mathbb{N}$, so $(x+1)^2 \geq y$. But also $(x+1)^2 \neq y$, since $2x \neq 0$. Hence $(x+1)^2 > y$.

10. Let p, q be propositions. Find a compound proposition, using the operator nand ($\uparrow$) exactly three times (and no other operators), that is logically equivalent to $p \vee q$. Prove your answer.

The desired proposition is $(p \uparrow p) \uparrow (q \uparrow q)$; its equivalence to $p \vee q$ is proved by the agreement of the fifth and sixth columns of the truth table at right.
Note: this was (part of) exercise 2.17.

p	q	$p \uparrow p$	$q \uparrow q$	$(p \uparrow p) \uparrow (q \uparrow q)$	$p \vee q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F